

# RATIONAL DISCRETE FIRST DEGREE COHOMOLOGY FOR TOTALLY DISCONNECTED LOCALLY COMPACT GROUPS

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ABSTRACT. It is well-known that the existence of more than two ends in the sense of J.R. Stallings for a finitely generated discrete group  $G$  can be detected on the cohomology group  $H^1(G, R[G])$ , where  $R$  is either a finite field, the ring of integers or the field of rational numbers. It will be shown (cf. Theorem A\*) that for a compactly generated totally disconnected locally compact group  $G$  the same information about the number of ends of  $G$  in the sense of H. Abels can be provided by  $dH^1(G, Bi(G))$ , where  $Bi(G)$  is the rational discrete standard bimodule of  $G$ , and  $dH^\bullet(G, \underline{\phantom{x}})$  denotes rational discrete cohomology as introduced in [6].

As a consequence one has that the class of fundamental groups of a finite graph of profinite groups coincides with the class of compactly presented totally disconnected locally compact groups of rational discrete cohomological dimension at most 1 (cf. Theorem B).

## 1. INTRODUCTION

For a totally disconnected locally compact (= t.d.l.c.) group  $G$  several cohomology theories can be introduced, e.g., the Bredon cohomology with respect to the family of all compact open subgroups of  $G$  and the continuous cohomology via cochain complexes. In this paper we investigate the rational discrete first degree cohomology of a t.d.l.c. group  $G$  as introduced in [6]. In Remarks 3.12 and 3.11 we provide a brief comparison of this cohomology theory with Bredon and continuous cohomology, respectively.

A left  $\mathbb{Q}[G]$ -module  $M$  is said to be *discrete* if the map  $\underline{\cdot} \cdot \underline{\phantom{x}}: G \times M \rightarrow M$  is continuous, where  $M$  carries the discrete topology. The category  $_{\mathbb{Q}[G]}\mathbf{dis}$  of discrete left  $\mathbb{Q}[G]$ -modules is an abelian category with both enough injectives and projectives. The right derived functors of  $\text{Hom}_{\mathbb{Q}[G]}(\underline{\phantom{x}}, \underline{\phantom{x}})$  have been denoted by  $d\text{Ext}_G^\bullet(\underline{\phantom{x}}, \underline{\phantom{x}})$ , and, for any  $k \geq 0$ , the group

$$dH^k(G, \underline{\phantom{x}}) = d\text{Ext}^k(\mathbb{Q}, \underline{\phantom{x}})$$

is defined to be the  $k^{\text{th}}$  *rational discrete cohomology group* of  $G$  with coefficients in  $_{\mathbb{Q}[G]}\mathbf{dis}$  (a brief introduction to this cohomology theory and some properties we use further on are given in §3.1).

In this paper we provide several results on the low-dimensional rational discrete cohomology of  $G$  by analogy with the discrete case. In section 3, we

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show that the functor  $dH^1(G, \underline{\phantom{x}})$  can be described by means of continuous derivations (cf. Propositions 3.9), and consequently by the almost invariant functions when we consider coefficients in a discrete permutation module (cf. Proposition 3.15).

In section 4, we prove the first main theorem of this paper (cf. Theorem A\*), which provides a cohomological interpretation of Stallings' decomposition theorem for compactly generated t.d.l.c. groups (cf. Theorem A). A compactly generated t.d.l.c. group  $G$  is said to *split non-trivially over a compact open subgroup  $K$*  if one of the following holds:

- (S1)  $G$  is a free product with amalgamation  $H *_K J$ , where  $H$  and  $J$  are compactly generated open subgroups satisfying  $K \neq H$  and  $K \neq J$ ;
- (S2)  $G$  is a HNN-extension  $H *_K^t$  with stable letter  $t$ , where  $H$  is a compactly generated open subgroup of  $G$ .

The space of *rough ends* of a compactly generated t.d.l.c. group  $G$  is defined to be the end space of a rough Cayley graph of  $G$  (cf. [17, §3] and §2.2). Thus the analogue of Stallings' decomposition theorem for t.d.l.c. groups can be restated as follows.

**Theorem A** ([17, Theorem 13]). *Let  $G$  be a compactly generated t.d.l.c. group, and let  $e(G)$  denote the number of rough ends of  $G$ . Then the following are equivalent:*

- (a)  $e(G) > 1$ , i.e.,  $G$  has more than one rough end;
- (b)  $G$  splits non-trivially over a compact open subgroup.

This splitting theorem is essentially due to Abels [1, Struktursatz 5.7, Korollar 5.8] and [17, §3.6] explains the relation with Abels' work in detail. In particular, it has been shown that the ideal points of the Specker compactification of a compactly generated t.d.l.c. group  $G$  can be identified with the rough ends of  $G$ , which definition here is recalled in §2.2.

The main purpose of this paper is to give the following cohomological reformulation of Theorem A.

**Theorem A\*.** *Let  $G$  be a compactly generated t.d.l.c. group. Then (a) and (b) of Theorem A are equivalent to*

$$(c) \quad dH^1(G, Bi(G)) \neq 0.$$

Here  $Bi(G)$  denote the *rational discrete standard bimodule* introduced in [6] to be a suitable substitute of the group algebra  $\mathbb{Q}[G]$  in the context of rational discrete cohomology. The rational discrete standard bimodule is defined by

$$Bi(G) = \varinjlim_{\mathcal{O} \in \mathfrak{CO}(G)} (\mathbb{Q}[G/\mathcal{O}], \eta_{U,V}), \quad (1.1)$$

where  $\mathfrak{CO}(G) = \{ \mathcal{O} \subset G \mid \mathcal{O} \text{ compact open subgroup} \}$ , and the direct limit is taken along the injective mappings

$$\eta_{U,V}: \mathbb{Q}[G/U] \rightarrow \mathbb{Q}[G/V], \quad \eta_{U,V}(gU) = \frac{1}{|U : V|} \sum_{r \in \mathcal{R}} grV, \quad g \in G \quad (1.2)$$

where  $V \subset U$  are compact open subgroups of  $G$  and  $\mathcal{R}$  is a set of coset representatives of  $U/V$ .

Now the new condition (c) guarantees a non-trivial splitting of a compactly generated t.d.l.c. group by knowing a single cohomology group as [8, Theorem IV 6.10] guarantees for finitely generated discrete groups.

The presence of the cohomological condition (c) leads us to prove Theorem A\* by means of the chain of implications  $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$ . Clearly,  $(a) \Rightarrow (b)$  has been already proven in Theorem A. Nevertheless, we prefer to provide a similar but substantially different proof (cf. Remark 4.3). Moreover, we obtain a new proof for  $(b) \Rightarrow (a)$  going through (c) that clarify how the three different aspects of a compactly generated t.d.l.c. group encoded in the conditions (a),(b) and (c) are related.

Stallings' theory of ends for discrete groups had certainly a major impact on geometric group theory. For example, his decomposition theorem - together with Bass-Serre's theory of groups acting on trees - was an essential tool for proving important results on groups of (virtual) cohomological dimension 1 like the Stallings-Swan theorem (cf. [22, 23]) or the Karrass-Pietrowski-Solitar theorem (cf. [15]). In particular, Stallings' decomposition theorem led naturally to the accessibility problem for finitely generated groups. Within the framework of Bass-Serre theory, a finitely generated group is said to be *accessible* if it is isomorphic to a fundamental group of a finite graph of groups such that every edge group is finite and every vertex group is a finitely generated group with at most one end. Equivalently, a compactly generated t.d.l.c. group can be defined to be *accessible* if it has an action on a tree such that

- (A1) the number of the  $G$ -orbits on the edges is finite;
- (A2) the edge-stabilizers are compact open subgroups of  $G$ ;
- (A3) every vertex-stabilizer is a compactly generated open subgroup of  $G$  with at most one rough end.

In 1991 M.J. Dunwoody [11] constructed an inaccessible finitely generated (discrete) group with infinitely many ends. In [17] the authors related the accessibility of a compactly generated t.d.l.c. group  $G$  to the accessibility of some (and hence all) rough Cayley graph of  $G$ , which is the analogue of [24, Theorem 1.1]. In 1985 M.J. Dunwoody [10] proved that every finitely presented (discrete) group has to be accessible. The analogue of this result in the context of t.d.l.c. groups is due to Y. Cornulier [7]. By using this accessibility result, we prove the second main theorem of this paper (cf. Theorem B).

**Theorem B.** *For every t.d.l.c. group  $G$ , the following are equivalent:*

- (i) *the group  $G$  is a compactly presented t.d.l.c. group with rational discrete cohomological dimension less or equal to one,*
- (ii) *the group  $G$  is isomorphic to the fundamental group  $\pi_1(\mathcal{G}, \Lambda)$  of a finite graph of profinite groups  $(\mathcal{G}, \Lambda)$ .*

This result is the (compactly presented) analogue of [9, Theorem. 1.1] that characterizes the (discrete) groups of cohomological dimension at most 1 over a commutative ring  $R$  to be fundamental groups of graphs of finite groups with no  $R$ -torsion.

**Question 1.** *Is every compactly generated t.d.l.c. group of rational discrete cohomological dimension at most 1 isomorphic to a fundamental group of a graph of profinite groups?*

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## 2. PRELIMINARIES ON ENDS

**2.1. Graphs.** In this paper we use the notion of *graph* as introduced by J-P. Serre in [21], i.e., a graph  $\Gamma$  consists of a set  $V(\Gamma)$ , a set  $E(\Gamma)$  and two maps

$$\begin{aligned} E(\Gamma) &\rightarrow V(\Gamma) \times V(\Gamma) & \mathbf{e} &\mapsto (o(\mathbf{e}), t(\mathbf{e})), \\ E(\Gamma) &\rightarrow E(\Gamma) & \mathbf{e} &\mapsto \bar{\mathbf{e}}, \end{aligned}$$

satisfying the following condition: for each  $\mathbf{e} \in E(\Gamma)$  we have  $\bar{\bar{\mathbf{e}}} = \mathbf{e}$ ,  $\bar{\mathbf{e}} \neq \mathbf{e}$  and  $o(\mathbf{e}) = t(\bar{\mathbf{e}})$ . An element  $v \in V(\Gamma)$  is called a *vertex* of  $\Gamma$ ; an element  $\mathbf{e} \in E(\Gamma)$  is called an *(oriented) edge* and  $\bar{\mathbf{e}}$  is its *inverse edge*. The 2-set  $\{\mathbf{e}, \bar{\mathbf{e}}\}$  is called a *geometric edge* of  $\Gamma$ . The vertex  $o(\mathbf{e})$  is called the *origin* of  $\mathbf{e}$  and the vertex  $t(\mathbf{e})$  is called the *terminus* of  $\mathbf{e}$ . A *path* from a vertex  $v$  to a vertex  $w$  in  $\Gamma$  is defined to be a sequence of edges  $\mathbf{p} = (\mathbf{e}_i)_{1 \leq i \leq r}$  such that  $o(\mathbf{e}_1) = v, t(\mathbf{e}_r) = w$  and  $t(\mathbf{e}_i) = o(\mathbf{e}_{i+1})$  for  $i = 1, \dots, r-1$ . A path  $\mathbf{p} = (\mathbf{e}_i)_{1 \leq i \leq r}$  is said to be *reduced* if  $\mathbf{e}_i \neq \bar{\mathbf{e}}_{i+1}$  for every  $i = 1, \dots, r-1$ . A reduced path  $\mathbf{p} = (\mathbf{e}_i)_{1 \leq i \leq r}$  satisfying  $t(\mathbf{e}_r) = o(\mathbf{e}_1)$  is called *circuit of length r*, and a *loop* is a circuit of length 1. A graph  $\Gamma$  is said to be *connected*, if there exists a path from any vertex  $v$  to any other vertex  $w$ . Every connected subgraph of  $\Gamma$  which is maximal with respect to this property is called a *connected component* of  $\Gamma$ . Thus every graph  $\Gamma$  is the disjoint union of its connected components and in this way one defines an equivalence relation  $\sim$  on  $V(\Gamma)$ , which is called the *connectedness relation*. A connected non-empty graph without circuits is said to be a *tree*.

For a graph  $\Gamma$  we denote by  $\underline{V}(\Gamma)$  the free  $\mathbb{Q}$ -vector space  $\mathbb{Q}[V(\Gamma)]$  over the set of vertices. If  $\mathbb{Q}[E(\Gamma)]$  denotes the  $\mathbb{Q}$ -vector space over the set  $E(\Gamma)$  we put

$$\underline{E}(\Gamma) = \mathbb{Q}[E(\Gamma)] / \text{span}_{\mathbb{Q}}\{\mathbf{e} + \bar{\mathbf{e}} \mid \mathbf{e} \in E(\Gamma)\} \quad (2.1)$$

the  $\mathbb{Q}$ -vector space freely generated by the geometric edges of  $\Gamma$ . Then one has the canonical  $\mathbb{Q}$ -linear map  $\delta : \underline{E}(\Gamma) \rightarrow \underline{V}(\Gamma)$  given by

$$\delta([\mathbf{e}]) = t(\mathbf{e}) - o(\mathbf{e}), \quad \mathbf{e} \in E(\Gamma), \quad (2.2)$$

where  $[\mathbf{e}]$  denotes the canonical image of  $\mathbf{e} \in E(\Gamma)$  in  $\underline{E}(\Gamma)$ . Let  $H_{\bullet}(|\Gamma|; \mathbb{Q})$  denote the singular homology groups with rational coefficients of the topological realization  $|\Gamma|$  of  $\Gamma$ . One has the following well known result.

**Fact 2.1** ([21, Corollary 1]). *Let  $\Gamma$  be a graph and let  $\delta : \underline{E}(\Gamma) \rightarrow \underline{V}(\Gamma)$  be the map given by (2.2). Then*

$$(a) \quad \ker(\delta) \cong H_1(|\Gamma|; \mathbb{Q}).$$

(b)  $\text{coker}(\delta) \cong \mathbb{Q}[\text{V}(\Gamma)/\sim]$ , where  $\sim$  is the connectedness relation.

In particular,  $\Gamma$  is a tree if, and only if,  $\ker(\delta) = 0$  and  $\text{coker}(\delta) \cong \mathbb{Q}$ .

Thus, given a connected graph  $\Gamma$ , one has an associated exact sequence

$$0 \longrightarrow \text{H}_1(|\Gamma|; \mathbb{Q}) \longrightarrow \underline{\text{E}}(\Gamma) \xrightarrow{\delta} \underline{\text{V}}(\Gamma) \longrightarrow \mathbb{Q} \longrightarrow 0 \quad (2.3)$$

of  $\mathbb{Q}$ -vector spaces.

**Example 2.2.** Let  $G$  be a t.d.l.c. group, and let  $\mathbb{Q}[G]\text{dis}$  denote the abelian category whose objects are the discrete left  $\mathbb{Q}[G]$ -modules (i.e., left  $\mathbb{Q}[G]$ -modules such that the pointwise stabilizers are open subgroups of  $G$ ).

(a) Suppose there exist open subgroups  $H, K$  and  $J$  such that  $G = H *_K J$ , i.e.,  $G$  splits as free product with amalgamation in  $K$ . The group  $G$  is then acting discretely - i.e. with open vertex stabilizers - without edge inversions on a tree with a segment as fundamental domain (cf. [21, Theorem 6]). By applying the orbit-stabilizer theorem, the exact sequence (2.3) yields

$$0 \longrightarrow \mathbb{Q}[G/K] \xrightarrow{\delta} \mathbb{Q}[G/H] \oplus \mathbb{Q}[G/J] \longrightarrow \mathbb{Q} \longrightarrow 0,$$

which is a short exact sequence in  $\mathbb{Q}[G]\text{dis}$ .

(b) Suppose  $G = H *_K^t$  is an HNN-extension with stable letter  $t$ , where  $H, K$  are open subgroups of  $G$ . Thus  $G$  is acting discretely and without edge inversions on a tree with a loop as fundamental domain (cf. [21, Remark 1, pg. 34]). Thus one has the following short exact sequence in  $\mathbb{Q}[G]\text{dis}$

$$0 \longrightarrow \mathbb{Q}[G/K] \xrightarrow{\delta} \mathbb{Q}[G/H] \longrightarrow \mathbb{Q} \longrightarrow 0.$$

**2.2. The number of rough ends.** A graph  $\Gamma$  is said to be *locally finite* if the set

$$\text{star}_\Gamma(v) = \{\mathbf{e} \in \text{E}(\Gamma) \mid o(\mathbf{e}) = v\}$$

is finite for every  $v \in \text{V}(\Gamma)$ . From now on  $\Gamma$  will be a connected locally finite graph. For a finite subset  $S \subseteq \text{V}(\Gamma)$  let  $E_S(\Gamma) = \{\mathbf{e} \in \text{E}(\Gamma) \mid o(\mathbf{e}) \in S\}$ , i.e., the union of all  $\text{star}_\Gamma(v)$ ,  $v \in S$ . We denote by  $\Gamma - S$  the subgraph of  $\Gamma$  with vertex set  $\text{V}(\Gamma) - S$  and edge set  $\text{E}(\Gamma) - (E_S(\Gamma) \cup \overline{E_S(\Gamma)})$ , i.e.,  $\Gamma - S$  is the subgraph obtained from  $\Gamma$  by removing  $S$  and all the edges attached to  $S$ . Let  $c_S$  be the number of infinite connected components of  $\Gamma - S$ . For a connected locally finite graph  $\Gamma$

$$e(\Gamma) = \sup\{c_S \mid S \subset \text{V}(\Gamma) \text{ finite}\} \quad (2.4)$$

will be called the *number of ends* of  $\Gamma$ . In particular, the graph  $\Gamma$  is finite if, and only if,  $\Gamma$  is 0-ended.

**Fact 2.3.** The number  $\text{E}(\Gamma)$  is greater than one if, and only if, there exists an infinite connected subgraph  $\mathcal{C} \subset \Gamma$  such that the set

$$\delta\mathcal{C} = \{\mathbf{e} \in \text{E}(\Gamma) \mid \text{either } o(\mathbf{e}) \in V(\mathcal{C}) \text{ or } t(\mathbf{e}) \in V(\mathcal{C}) \text{ but not both}\} \quad (2.5)$$

is finite and the subgraph  $\mathcal{C}^* = \Gamma - V(\mathcal{C})$  contains an infinite connected component.

The set of vertices  $C = V(\mathcal{C})$  is called a *cut* of  $\Gamma$ .

Recall that two connected graphs  $(\Gamma, d_\Gamma)$  and  $(\Gamma', d_{\Gamma'})$  (with the geodesic metric) are said to be *quasi-isometric* if there exist a map  $\varphi : V(\Gamma) \rightarrow V(\Gamma')$  and constants  $a \geq 1$  and  $b > 0$  such that for all vertices  $v, w \in V(\Gamma)$

$$a^{-1}d_\Gamma(v, w) - a^{-1}b \leq d_{\Gamma'}(\varphi(v), \varphi(w)) \leq a d_\Gamma(v, w), \quad (2.6)$$

and for all vertices  $v' \in V(\Gamma')$  one has

$$d_\Gamma(v', \varphi(V(\Gamma))) \leq b. \quad (2.7)$$

A map  $\varphi$  satisfying the above conditions is called a *quasi-isometry* of graphs. Moreover, the relation of being quasi-isometric is an equivalence relation among graphs and the number of ends is a quasi-isometric invariant (cf. [18, Proposition 1]).

A t.d.l.c. group  $G$  is said to be *compactly generated* if there exist a compact open subgroup  $K$  and a finite symmetric set  $S \subset G \setminus K$  such that  $G$  is algebraically generated by  $S \cup K$ . Every such a pair  $(K, S)$  will be called a *generating pair* of  $G$ . The *rough Cayley graph*  $\Gamma$  associated to  $G$  with respect to the generating pair  $(K, S)$  consists of the following data:

$$V(\Gamma) = G/K, \quad E(\Gamma) = \{(gK, gsK), (gsK, gK) \mid g \in G, s \in S\}, \quad (2.8)$$

where the origin and terminus maps are given by projection onto the first and second coordinate, respectively, while the edge inversion mapping permutes the first and second coordinate.

**Remark 2.4.** *In literature these graphs are also known as Cayley-Abels graphs. The definition we have chosen here follows the approach used in [17, §2], with the difference that the edges of a graph are directed in our setup.*

A rough Cayley graph  $\Gamma$  is naturally endowed with a discrete  $G$ -action, i.e.,  $G$  is acting with open stabilizers. Moreover, the following fact holds.

**Fact 2.5.** *Let  $G$  be a compactly generated t.d.l.c. group. Then*

- (a) *every rough Cayley graph  $\Gamma$  of  $G$  is a vertex-transitive, connected and locally finite graph;*
- (b)  *$G$  has a continuous, proper and cocompact  $G$ -action on  $\Gamma$ ;*
- (c) *all rough Cayley graphs of  $G$  are quasi-isometric;*
- (d) *all rough Cayley graphs of  $G$  have the same number of ends.*

Thus the *number of rough ends*  $e(G)$  of a compactly generated t.d.l.c. group  $G$  can be defined to be the number of ends of a rough Cayley graph  $\Gamma$  associated to  $G$  with respect to some generating pair  $(K, S)$ .

**Example 2.6.** (a) *If  $G$  is a finitely generated discrete group, then the notion of rough Cayley graph gives back the well-known notion of Cayley graph and its number of ends. E.g.  $\mathbb{Z}$  and  $D_\infty$  are 2-ended groups.*

(b) *The group  $SL_2(\mathbb{Q}_p)$  is a free product with amalgamation of two copies of  $SL_2(\mathbb{Z}_p)$ . Hence  $SL_2(\mathbb{Q}_p)$  has infinitely many rough ends.*

## 3. FIRST DEGREE COHOMOLOGY

**3.1. Rational discrete cohomology.** Here we collect some of the properties concerning the rational discrete cohomology for t.d.l.c. groups we shall use further on. For the details the reader is referred to [6].

For a t.d.l.c. group  $G$ , let  $\mathbb{Q}[G]\mathbf{dis}$  denote the abelian full subcategory of  $\mathbb{Q}[G]\mathbf{mod}$  whose objects are the discrete left  $\mathbb{Q}[G]$ -modules, i.e., left  $\mathbb{Q}[G]$ -modules with open stabilizers. The category  $\mathbb{Q}[G]\mathbf{dis}$  has enough injectives, thus one may define

$$\mathrm{dExt}_G^k(M, -) = \mathcal{R}^k \mathrm{Hom}_{\mathbb{Q}[G]\mathbf{dis}}(M, -) \quad (3.1)$$

the right derived functors of  $\mathrm{Hom}_{\mathbb{Q}[G]}(M, -)$  in  $\mathbb{Q}[G]\mathbf{dis}$ , and the  $k^{\text{th}}$  discrete cohomology group of  $G$  with coefficients in  $\mathbb{Q}[G]\mathbf{dis}$  by

$$\mathrm{dH}^k(G, -) = \mathrm{dExt}_G^k(\mathbb{Q}, -), \quad k \geq 0, \quad (3.2)$$

where  $\mathbb{Q}$  denotes the trivial discrete left  $\mathbb{Q}[G]$ -module.

By using Maschke's theorem, one may prove that the trivial  $\mathbb{Q}[G]$ -module  $\mathbb{Q}$  is projective whenever  $G$  is profinite. Consequently, for every t.d.l.c. group  $G$ , the discrete left  $\mathbb{Q}[G]$ -module  $\mathbb{Q}[G/K]$  is projective in  $\mathbb{Q}[G]\mathbf{dis}$  whenever  $K$  is a compact open subgroup of  $G$ . Moreover, one may stress further this property as follows.

Let  $\Omega$  be a left  $G$ -set with open point-wise stabilizers. Clearly,  $\mathbb{Q}[\Omega]$  - the free  $\mathbb{Q}$ -vector space over the set  $\Omega$  - is a discrete left  $\mathbb{Q}[G]$ -module, which is also called a *discrete left  $\mathbb{Q}[G]$ -permutation module*.

**Proposition 3.1** ([6, Prop. 3.2]). *Let  $G$  be a t.d.l.c. group, and let  $\Omega$  be a left  $G$ -set with compact open stabilizers. Then  $\mathbb{Q}[\Omega]$  is projective in  $\mathbb{Q}[G]\mathbf{dis}$ . In particular, the abelian category  $\mathbb{Q}[G]\mathbf{dis}$  has enough projectives.*

The existence of projective resolutions in  $\mathbb{Q}[G]\mathbf{dis}$  naturally leads to several finiteness conditions on  $G$  as usual. Firstly, the *rational discrete cohomological dimension* of  $G$ , denoted by  $\mathrm{cd}_{\mathbb{Q}}(G)$ , is defined to be the smallest non-negative integer  $n$  such that there exists a projective resolution  $(P_i, \partial_i)$  of  $\mathbb{Q}$  in  $\mathbb{Q}[G]\mathbf{dis}$  of length  $\leq n$ . Analogously to the discrete case, one has the following properties.

**Proposition 3.2** ([6, Prop. 3.7]). *Let  $G$  be a t.d.l.c. group.*

- (a)  *$G$  is compact if, and only if,  $\mathrm{cd}_{\mathbb{Q}}(G) = 0$ .*
- (b) *If  $H$  is a closed subgroup of  $G$ , then*

$$\mathrm{cd}_{\mathbb{Q}}(H) \leq \mathrm{cd}_{\mathbb{Q}}(G).$$

Moreover, a discrete left  $\mathbb{Q}[G]$ -module  $M$  is said to be *finitely generated*, if there exist a finite number of compact open subgroups  $K_1, \dots, K_n$  of  $G$  and an epimorphism  $\pi: \coprod_{1 \leq j \leq n} \mathbb{Q}[G/K_j] \rightarrow M$ . Consequently, a discrete left  $\mathbb{Q}[G]$ -module  $M$  is said to be of type  $\mathrm{FP}_n$ ,  $n \geq 0$ , if  $M$  satisfies one of the following equivalent properties:

- (F1) there is a partial projective resolution

$$P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

of  $M$  in  $\mathbb{Q}[G]\mathbf{dis}$  such that  $P_j$  is finitely generated for all  $0 \leq j \leq n$ ;

(F2)  $M$  is finitely generated and for every partial projective resolution

$$Q_k \xrightarrow{\partial_k} Q_{k-1} \xrightarrow{\partial_{k-1}} \dots \xrightarrow{} Q_1 \xrightarrow{} Q_0 \xrightarrow{} M \xrightarrow{} 0$$

in  $\mathbb{Q}[G]\mathbf{dis}$  with  $k < n$  such that  $Q_j$  is finitely generated for all  $j = 0, \dots, k$ , one has that  $\ker(\partial_k)$  is finitely generated.

E.g.,  $M$  is of type  $\text{FP}_0$  if, and only if,  $M$  is finitely generated. If  $M$  is of type  $\text{FP}_n$  for all  $n \geq 0$ , then  $M$  is called to be of type  $\text{FP}_\infty$ . Accordingly, the group  $G$  is said to be of type  $\text{FP}_n$ ,  $n \in \mathbb{N} \cup \{\infty\}$ , if the trivial module  $\mathbb{Q}$  is of type  $\text{FP}_n$  in  $\mathbb{Q}[G]\mathbf{dis}$ .

**Proposition 3.3.** *Let  $G$  be a t.d.l.c. group and  $A \in \text{ob}(\mathbb{Q}[G]\mathbf{dis})$  of type  $\text{FP}_n$ ,  $n \geq 0$ . Then for every direct limit  $\varinjlim M_\bullet$  in  $\mathbb{Q}[G]\mathbf{dis}$  the natural homomorphism*

$$\varinjlim \text{dExt}_G^k(A, M_\bullet) \rightarrow \text{dExt}_G^k(A, \varinjlim M_\bullet),$$

*is an isomorphism for  $k \leq n-1$  and a monomorphism for  $k = n$ .*

*Proof.* Let  $(P_\bullet, \partial_\bullet, \epsilon)$  be a projective resolution of  $\mathbb{Q}$  in  $\mathbb{Q}[G]\mathbf{dis}$  such that  $P_j$  is finitely generated for  $0 \leq j \leq n$ . By the  $\text{Hom} - \otimes$  identity provided in [6, §4.3],  $\text{Hom}_G(P_j, -)$  commutes with direct limits whenever  $0 \leq j \leq n$ . Thus the proof of [4, Prop. 1.2] can be transferred here.  $\square$

**Remark 3.4.** *If all of the canonical maps  $M_\bullet \rightarrow \varinjlim M_\bullet$  are injective, then an easy diagram chasing shows that  $\varinjlim \text{dExt}_G^n(A, M_\bullet) \rightarrow \text{dExt}^n(A, \varinjlim M_\bullet)$  is an isomorphism as well.*

**Corollary 3.5.** *For a t.d.l.c. group  $G$  of type  $\text{FP}_\infty$  the functors  $\text{dH}^\bullet(G, -)$  commute with direct limits in  $\mathbb{Q}[G]\mathbf{dis}$ .*

It is well known that a discrete group is finitely generated if, and only if, it is of type  $\text{FP}_1$  (cf. [5, §VIII.4]). The analogue result for t.d.l.c. groups holds as well.

**Proposition 3.6** ([6, Prop. 5.3]). *Let  $G$  be a t.d.l.c. group. Then  $G$  is compactly generated if, and only if,  $G$  is of type  $\text{FP}_1$ .*

By combining the latter finiteness conditions, one defines a t.d.l.c. group  $G$  to be of type  $\text{FP}$ , if  $G$  is of type  $\text{FP}_\infty$  with  $\text{cd}_{\mathbb{Q}}(G) = d < \infty$ . In other words, the trivial left  $\mathbb{Q}[G]$ -module  $\mathbb{Q}$  has a projective resolution which is finitely generated and concentrated in degrees 0 to  $d$ .

Since the group algebra  $\mathbb{Q}[G]$  is not a discrete  $\mathbb{Q}[G]$ -module unless the group  $G$  itself is discrete, in [6] a possible substitute has been introduced and studied. Namely, the rational discrete standard bimodule  $\text{Bi}(G)$  (cf. (1.1)). The following are in analogy with the discrete case.

**Fact 3.7** ([6, Prop. 4.3]). *Let  $G$  be a t.d.l.c. group. One has*

$$\text{Hom}_G(\mathbb{Q}, \text{Bi}(G)) \simeq \begin{cases} \mathbb{Q} & \text{if } G \text{ is compact,} \\ 0 & \text{if } G \text{ is not compact.} \end{cases}$$

**Proposition 3.8** ([6, Prop. 4.7]). *Let  $G$  be a t.d.l.c. group of type  $\text{FP}$ . Then*

$$\text{cd}_{\mathbb{Q}}(G) = \max\{k \geq 0 \mid \text{dH}^k(G, \text{Bi}(G)) \neq 0\}. \quad (3.3)$$

**3.2. Derivations.** Let  $\text{Der}(G, M)$  denote the group of all (algebraic) derivations  $d$  from a group  $G$  to a left  $G$ -module  $M$ , i.e.,  $d$  is a mapping of sets  $d: G \rightarrow M$  satisfying  $d(gh) = gd(h) + d(g)$  for all  $g, h \in G$ .

For a t.d.l.c. group  $G$  and a discrete  $\mathbb{Q}[G]$ -module  $M$ , we define

$$\text{Der}_K(G, M) = \{ d \in \text{Der}(G, M) \mid d(k) = 0, \forall k \in K \}, \quad (3.4)$$

$$\text{PDer}_K(G, M) = \{ d \in \text{Der}_K(G, M) \mid \exists m \in M^K \text{ s.t. } d(g) = gm - m \forall g \in G \},$$

where  $K$  is a compact open subgroup of  $G$ . Clearly every element  $d$  of  $\text{Der}_K(G, M)$  is a continuous map, where  $M$  carries the discrete topology.

By analogy to the discrete case, one may prove the following result and we include the standard proof for reader's convenience.

**Proposition 3.9.** *For a compact open subgroup  $K$  of a t.d.l.c. group  $G$  there is a natural isomorphism*

$$\text{dH}^1(G, M) \cong \text{Der}_K(G, M)/\text{PDer}_K(G, M),$$

where  $M \in \text{ob}(\mathbb{Q}[G]\text{dis})$ .

*Proof.* Let

$$0 \longrightarrow N \longrightarrow \mathbb{Q}[G/K] \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0 \quad (3.5)$$

be the short exact sequence in  $\mathbb{Q}[G]\text{dis}$  provided by the augmentation map  $\varepsilon$ . Thus the set  $\{gK - K \mid g \in G \setminus K\}$  is a generating set of  $N$  as  $\mathbb{Q}$ -vector space. Firstly, notice that

$$\text{Hom}_G(N, M) \cong \text{Der}_K(G, M),$$

for every  $M \in \text{ob}(\mathbb{Q}[G]\text{dis})$ . Indeed for every  $\mathbb{Q}[G]$ -map  $\varphi: N \rightarrow M$  let

$$D_\varphi: G \rightarrow M, \quad D_\varphi(g) = \varphi(gK - K) \quad \forall g \in G. \quad (3.6)$$

Clearly,  $D_\varphi \in \text{Der}_K(G, M)$ . Thus the formula (3.6) defines a natural homomorphism from  $\text{Hom}_G(N, M)$  to  $\text{Der}_K(G, M)$ . This homomorphism admits the inverse  $D \mapsto \varphi_D$  given by  $\varphi_D(gK - K) = D(g)$ , which is well-defined since  $D \in \text{Der}_K(G, M)$  is constant on the cosets of  $K$  in  $G$ .

By applying the long exact cohomology functor to (3.5) with coefficients in  $M$ , one has

$$0 \longrightarrow M^G \longrightarrow M^K \longrightarrow \text{Der}_K(G, M) \longrightarrow \text{dH}^1(G, M) \longrightarrow 0, \quad (3.7)$$

since  $\mathbb{Q}[G/K]$  is projective in  $\mathbb{Q}[G]\text{dis}$  and  $\text{Hom}_G(\mathbb{Q}[G/K], M) \cong \text{Hom}_K(\mathbb{Q}, M)$  (cf. Proposition 3.1 and [6, §2.9]). Finally, as  $\text{PDer}_K(G, M) \cong M^K/M^G$  by definition, (3.7) yields the claim.  $\square$

**Corollary 3.10.** *For a t.d.l.c. group  $G$  and  $M \in \text{ob}(\mathbb{Q}[G]\text{dis})$ , let  $\text{Der}_{\text{top}}(G, M)$  be the group of all continuous derivations from  $G$  to  $M$  and  $\text{PDer}_{\text{top}}(G, M)$  the subgroup of the principal one. Thus*

$$\text{dH}^1(G, M) \cong \text{Der}_{\text{top}}(G, M)/\text{PDer}_{\text{top}}(G, M),$$

naturally.

*Proof.* Let  $d$  be a continuous derivation from  $G$  to  $M$ . Then  $\rho: G \times M \rightarrow M$  given by  $\rho(g, m) = gm + d(g)$  defines a continuous affine transformation of  $M$ . For every compact open subgroup  $K$  of  $G$ , the  $K$ -orbit is finite, by continuity. So the average of this orbit is a  $K$ -fixed point, say  $x$ . Let

$d' \in \text{PDer}_K(G, M)$  be the principal derivation associated to  $x$ . Since  $d - d' \in \text{Der}_K(G, M)$ , every continuous 1-cocycle is cohomologous to a 1-cocycle vanishing on  $K$ .  $\square$

**Remark 3.11.** *One can chose to develop a cohomology theory for a t.d.l.c. group  $G$  directly via cochain complexes. For  $M \in \text{ob}(\mathbb{Q}[G]\text{dis})$  let  $C^n(G, M)$  be the set of all continuous functions from  $G^n$  to  $M$ , where  $M$  carries the discrete topology. By equipping this with the usual coboundary operators, one has a cochain complex whose cohomology can be defined to be the continuous cohomology of  $G$ , e.g. [13, 19]. By Corollary 3.10, the rational discrete cohomology of  $G$  turns out to be equivalent to the continuous one in degree 0 and 1, but at this stage we do not know if this is true for  $n \geq 2$ .*

**Remark 3.12.** *Let  $\mathcal{C}$  be the family of all compact open subgroups of a t.d.l.c. group  $G$ . By van Dantzig's Theorem,  $\mathcal{C}$  is non-empty. Furthermore  $\mathcal{C}$  is closed under conjugation and taking finite intersections. Let  $\mathcal{O}_{\mathcal{C}}(G)$  be the orbit category of  $G$  w.r.t.  $\mathcal{C}$ . Namely, the objects are the  $G$ -sets  $G/K$ , for  $K \in \mathcal{C}$ , and the morphisms are the  $G$ -maps between them. Thus one may define the category of Bredon modules over  $\mathcal{O}_{\mathcal{C}}(G)$  as usual. The Bredon cohomology of  $G$  is not equivalent to the rational discrete cohomology of  $G$ . Indeed a necessary condition for a t.d.l.c. group  $G$  to be of type  $\text{FP}_0$  in the Bredon cohomology is the following: there are finitely many compact open subgroups  $K_1, \dots, K_n$  of  $G$  such that any compact open subgroup of  $G$  is subconjugated to one of the  $K_i$ s (cf. [16, Lemma 2.3]). On the other hand, being of type  $\text{FP}_0$  for a t.d.l.c. group in the rational discrete cohomology is an empty condition.*

**Remark 3.13.** *We are aware of a possible connection between rational discrete cohomology and the cohomology of the Hecke algebra (cf. [20, §2]) but it will be not discussed in this paper.*

**3.3. The almost invariant functions.** In order to connect the rational discrete cohomology of  $G$  to the number of rough ends as clearly as possible, we provide another representation of  $\text{dH}^1(G, M)$  whenever  $M$  is a transitive discrete permutation module.

Let  $G$  be a compactly generated t.d.l.c. group and let  $(K, S)$  be a generating pair of  $G$ . Clearly, the set  $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}[G/K], \mathbb{Q})$  of all functions from  $G/K$  to  $\mathbb{Q}$  is a  $G$ -set with action given by

$$(g \cdot \alpha)(x) = \alpha(g^{-1}x) \quad \forall \alpha \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}[G/K], \mathbb{Q}), \quad \forall g \in G, \quad \forall x \in G/K. \quad (3.8)$$

Following [9], we say that two maps  $\alpha, \beta \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}[G/K], \mathbb{Q})$  are *almost equal*, and denote this by  $\alpha =_a \beta$ , if  $\alpha(x) = \beta(x)$  for all but finitely many elements  $x \in G/K$ .

**Example 3.14.** *Every element  $m \in \mathbb{Q}[G/K]$  can be expressed as formal sum*

$$m = \sum_{x \in G/K} q_x x$$

*with  $q_x \in \mathbb{Q}$  being 0 for almost all  $x \in G/K$ . Then  $m$  can be identified with the projection  $p_m : G/K \rightarrow \mathbb{Q}$  given by  $p_m(x) = q_x$ , showing that  $p_m =_a 0$ . Thus  $\mathbb{Q}[G/K]$  is the set of all almost zero functions in  $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}[G/K], \mathbb{Q})$ .*

An element  $\alpha \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}[G/K], \mathbb{Q})$  is called an *almost  $(G, K)$ -invariant function* if  $g \cdot \alpha =_a \alpha$  for all  $g \in G$  and  $k \cdot \alpha = \alpha$  for all  $k \in K$ . Denote by  $\mathcal{A}Inv_K(G, \mathbb{Q})$  the space of all almost  $(G, K)$ -invariant functions.

**Proposition 3.15.** *For every compact open subgroup  $K$  of a t.d.l.c. group  $G$  one has*

$$dH^1(G, \mathbb{Q}[G/K]) \cong \frac{\mathcal{A}Inv_K(G, \mathbb{Q})}{C(G/K) + \mathbb{Q}[G/K]^K},$$

where

$$C(G/K) = \{\alpha \in \text{Hom}(\mathbb{Q}[G/K], \mathbb{Q}) \mid \alpha \text{ constant}\},$$

and  $\mathbb{Q}[G/K]^K$  denotes the largest  $K$ -invariant submodule of  $\mathbb{Q}[G/K]$ .

*Proof.* The second part of the proof of Lemma 1.1 in [3] can be easily adapted to our context. Thus for every compact open subgroup  $K$  of  $G$  there exists the following short exact sequence

$$0 \longrightarrow C(G/K) \longrightarrow \mathcal{A}Inv_K(G, \mathbb{Q}) \xrightarrow{\partial} \text{Der}_K(G, \mathbb{Q}[G/K]) \longrightarrow 0, \quad (3.9)$$

where for each  $\alpha$  the map  $\partial\alpha: G \rightarrow \mathbb{Q}[G/K]$  is given by  $\partial\alpha(g) = g \cdot \alpha - \alpha$ .

As  $\text{PDer}_K(G, \mathbb{Q}[G/K]) \cong \mathbb{Q}[G/K]^K$ , applying Proposition 3.9 concludes the proof.  $\square$

#### 4. THE DECOMPOSITION THEOREM

The aim of this section is to prove Theorem A\*. Clearly, the proof of Theorem A\* can be shortened considering that the equivalence between a) and b) is well-known, but here we prove the result via the chain of implications  $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$ .

Recall that a t.d.l.c. group  $G$  acts *discretely* on a graph if the stabilizers are open subgroups of  $G$ .

**Proposition 4.1.** *Let  $G$  be a compactly generated t.d.l.c. group. Suppose that  $G$  acts discretely on a tree  $\mathcal{T}$  such that*

- (i) *the group  $G$  is acting without edge inversions;*
- (ii) *the quotient graph  $G \setminus \mathcal{T}$  is finite;*
- (iii) *the edge stabilizers  $G_e$  are compact open subgroups of  $G$ .*

*Then the vertex stabilizers  $G_v$  are compactly generated.*

*Proof.* Recall that a t.d.l.c. group is compactly generated if, and only if, it is of type  $\text{FP}_1$  (cf. Prop. 3.6). Thus it is sufficient to prove that the trivial module  $\mathbb{Q}$  is of type  $\text{FP}_1$  in  $\mathbb{Q}[G_v]\text{dis}$ , for all  $v \in V(\mathcal{T})$ . By property (i) and (2.3), one has that the following sequence

$$0 \longrightarrow \coprod_{e \in \mathcal{R}_E(\mathcal{T})} \mathbb{Q}[G/G_e] \longrightarrow \coprod_{v \in \mathcal{R}_V(\mathcal{T})} \mathbb{Q}[G/G_v] \longrightarrow \mathbb{Q} \longrightarrow 0 \quad (4.1)$$

is exact in  $\mathbb{Q}[G]\text{mod}$ , where  $\mathcal{R}_V(\mathcal{T})$  is a set of representatives of the  $G$ -orbits on  $V(\mathcal{T})$ , and  $\mathcal{R}_E(\mathcal{T})$  is a set of representatives of the  $C_2 \times G$ -orbits on  $E(\mathcal{T})$ . In particular,  $\mathcal{R}_V(\mathcal{T})$  and  $\mathcal{R}_E(\mathcal{T})$  are finite by (ii). Moreover,  $G$  is acting discretely on  $\mathcal{T}$ , i.e. with open stabilizers, thus (4.1) is a short exact sequence in  $\mathbb{Q}[G]\text{dis}$ . Thus one may consider the induction functors

$\text{ind}_{G_*}^G : \mathbb{Q}[G_*] \mathbf{dis} \rightarrow \mathbb{Q}[G] \mathbf{dis}$ , where  $* \in \{v, \mathbf{e} \mid v \in \mathcal{R}_V(\mathcal{T}), \mathbf{e} \in \mathcal{R}_E(\mathcal{T})\}$  (cf. [6, §2.4]). In particular (4.1) can be reformulated as follows

$$0 \longrightarrow \coprod_{\mathbf{e} \in \mathcal{R}_E(\mathcal{T})} \text{ind}_{G_{\mathbf{e}}}^G(\mathbb{Q}) \longrightarrow \coprod_{v \in \mathcal{R}_V(\mathcal{T})} \text{ind}_{G_v}^G(\mathbb{Q}) \longrightarrow \mathbb{Q} \longrightarrow 0. \quad (4.2)$$

For  $G$  is a compactly generated t.d.l.c. group, the trivial module  $\mathbb{Q}$  is of type  $\text{FP}_1$  in  $\mathbb{Q}[G] \mathbf{dis}$ . The permutation module  $\coprod_{\mathbf{e} \in \mathcal{R}_E(\mathcal{T})} \text{ind}_{G_{\mathbf{e}}}^G(\mathbb{Q})$  with compact open stabilizers is a finitely generated projective discrete  $\mathbb{Q}[G]$ -module, and so of type  $\text{FP}_1$  as well (cf. Proposition 3.1). By applying the horseshoe lemma to (4.2), one has that  $\coprod_{v \in \mathcal{R}_V(\mathcal{T})} \text{ind}_{G_v}^G(\mathbb{Q})$  is of type  $\text{FP}_1$  in  $\mathbb{Q}[G] \mathbf{dis}$ . Hence  $\text{ind}_{G_v}^G(\mathbb{Q})$  is of type  $\text{FP}_1$  for every  $v \in \mathcal{R}_V(\mathcal{T})$  (cf. [4, Prop. 1.4 (a)]). As the induction functor is exact and it is mapping projectives to projectives (cf. [6, Proposition 3.4]), one deduces that the trivial module  $\mathbb{Q}$  is of type  $\text{FP}_1$  in  $\mathbb{Q}[G_v] \mathbf{dis}$ , since  $\text{ind}_{G_v}^G(\mathbb{Q})$  is of type  $\text{FP}_1$  in  $\mathbb{Q}[G] \mathbf{dis}$  for every  $v \in \mathcal{R}_V(\mathcal{T})$ . By conjugation, (iv) holds.  $\square$

**Remark 4.2.** *It is possible to extend the previous result to actions with edge inversions. In such a case, one has to consider the stabilizers  $G_{\{\mathbf{e}\}}$  of the geometric edges  $\{\mathbf{e}, \bar{\mathbf{e}}\}$  (cf. [6, Prop. 5.4]). On the other hand, it is well-known that the condition about the action without edge-inversions is not properly a restriction, since it is always possible to consider the barycentric subdivision of the tree.*

*Proof of (a)  $\Rightarrow$  (b).* Starting from a rough Cayley graph associated to  $G$ , one may use different techniques to construct a tree satisfying the hypothesis in the previous result whenever  $G$  has more than one rough end (cf. [12], [8]). Thus the result follows by Proposition 4.1 and Bass-Serre theory.  $\square$

**Remark 4.3.** *In [17], to prove that a compactly generated t.d.l.c. group  $G$  with more than one rough end splits non-trivially over a compact open subgroup (namely, (a)  $\Rightarrow$  (b)) the authors applied the following technique. Firstly, by using the theory of structure trees developed in [8], they construct a directed tree acted on by  $G$  with finitely many orbits such that the edge stabilizers are compact and open and the vertex stabilizers are (open) subgroups of  $G$ . Secondly, they applied Bass-Serre theory of groups acting on trees to conclude that  $G$  has to split. Finally, they had to prove that every vertex stabilizer  $G_\alpha$  is compactly generated, which is the main part of the proof. They achieve this final step by constructing a connected locally finite graph acted on transitively by  $G_\alpha$  with compact open stabilizers (cf. [17, Theorem 1]). This graph is obtained by means of a construction developed in [24, Section 7]. By Proposition 4.1 instead, one directly deduces that the vertex stabilizers are compactly generated.*

*Proof of (b)  $\Rightarrow$  (c).* Let  $G$  split non-trivially over the compact open subgroup  $K$ , i.e., either (S1) or (S2) holds. The proof is split up as follows.

**Case 1.** According as the splitting type (i.e., either (S1) or (S2)), suppose  $H$  and  $J$  are both compact. By Bass-Serre's theory,  $G$  is acting on the universal covering tree  $\tilde{\Gamma}$ , thus (2.3) yields a short exact sequence

$$0 \longrightarrow \underline{E}(\tilde{\Gamma}) \xrightarrow{\delta} \underline{V}(\tilde{\Gamma}) \longrightarrow \mathbb{Q} \longrightarrow 0, \quad (4.3)$$

(cf. Example 2.2). Since the vertex stabilizers are conjugated to  $H$  (and  $J$  respectively),  $G$  is acting on  $\tilde{\Gamma}$  with compact open stabilizers. Hence (4.3) is a projective resolution of  $\mathbb{Q}$  of length 1 in  ${}_{\mathbb{Q}[G]}\mathbf{dis}$ , since it has discrete permutation  $\mathbb{Q}[G]$ -modules with compact stabilizers in degree 0 and 1 (cf. Proposition 3.1). Therefore  $\text{cd}_{\mathbb{Q}}(G) = 1$ , as  $G$  is non-compact (cf. Proposition 3.2(a)). By Proposition 3.6, since  $G$  is compactly generated,  $G$  is a t.d.l.c. group of type  $\text{FP}_1$  with  $\text{cd}_{\mathbb{Q}}(G) = 1$ , so  $G$  is of type FP. Thus Proposition 3.8 yields the claim.

**Case 2.** Assume  $G = H *_K^t$  and  $H$  is non-compact. As shown in Example 2.2(b), one has the following short exact sequence in  ${}_{\mathbb{Q}[G]}\mathbf{dis}$

$$0 \longrightarrow \mathbb{Q}[G/K] \xrightarrow{\delta} \mathbb{Q}[G/H] \longrightarrow \mathbb{Q} \longrightarrow 0. \quad (4.4)$$

Recall that for every compact open subgroup  $\mathcal{O}$  of  $G$  one has

$$\mathbb{Q}[G/\mathcal{O}] \cong \mathbb{Q}[G] \otimes_{\mathbb{Q}[\mathcal{O}]} \mathbb{Q} = \text{ind}_{\mathcal{O}}^G(\mathbb{Q}),$$

where  $\text{ind}_{\mathcal{O}}^G(-) : {}_{\mathbb{Q}[\mathcal{O}]}\mathbf{dis} \rightarrow {}_{\mathbb{Q}[G]}\mathbf{dis}$  is the induction functor (cf. [6, §2.4]). By the Eckmann-Shapiro type lemma [6, §2.9], applying the long exact cohomology functor with coefficients in  $\text{Bi}(G)$  yields the long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Bi}(G)^G & \longrightarrow & \text{Bi}(G)^H & \xrightarrow{\delta^*} & \text{Bi}(G)^K \\ & & & & & \downarrow & \\ & & & & & \text{dH}^1(G, \text{Bi}(G)) & \\ & & & & & \downarrow & \\ & & & & & \vdots & \end{array} \quad (4.5)$$

As  $H$  is not compact,  $\text{Bi}(G)^H = 0$  (cf. Fact 3.7). Thus (4.5) gives an injective map from  $\text{Bi}(G)^K$  to  $\text{dH}^1(G, \text{Bi}(G))$ . For  $K$  is compact,  $\text{Bi}(G)^K \neq 0$  and then  $\text{dH}^1(G, \text{Bi}(G)) \neq 0$ .

**Case 3.** Let  $G = H *_K J$  and  $H$  non-compact. The sequence

$$0 \longrightarrow \mathbb{Q}[G/K] \xrightarrow{\delta} \mathbb{Q}[G/H] \oplus \mathbb{Q}[G/J] \longrightarrow \mathbb{Q} \longrightarrow 0, \quad (4.6)$$

is exact in  ${}_{\mathbb{Q}[G]}\mathbf{dis}$  (cf. Example 2.2(a)).

Now for  $H$  is not compact, applying the long exact cohomology functor with coefficients in  $\mathbb{Q}[G/K]$  yields the long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Q}[G/K]^G & \longrightarrow & \text{Hom}_G(\mathbb{Q}[G/J], \mathbb{Q}[G/K]) & \xrightarrow{\delta^*} & \text{End}_G(\mathbb{Q}[G/K]) \\ & & & & & \downarrow & \\ & & & & & \text{dH}^1(G, \mathbb{Q}[G/K]) & \\ & & & & & \downarrow & \\ & & & & & \vdots & \end{array} \quad (4.7)$$

It follows that  $dH^1(G, \mathbb{Q}[G/K]) \neq 0$ . Indeed suppose firstly  $J$  to be non-compact. Thus Fact 3.7 implies that

$$\text{End}_G(\mathbb{Q}[G/K]) \rightarrow dH^1(G, \mathbb{Q}[G/K])$$

in (4.7) is injective.

On the other hand, if  $J$  is compact, we claim that  $\delta^*$  cannot be surjective, and so  $dH^1(G, \mathbb{Q}[G/K]) \neq 0$  as well.

Let us prove the claim. Recall that the map  $\delta$  in (4.7) is given by

$$\delta : \mathbb{Q}[G/K] \rightarrow \mathbb{Q}[G/H] \oplus \mathbb{Q}[G/J], \quad \delta(gK) = gH - gJ, \quad \forall g \in G.$$

Let  $\varphi \in \text{Hom}_G(\mathbb{Q}[G/J], \mathbb{Q}[G/K])$ , thus one has

$$\delta^*(\varphi)(g_1K) = \delta^*(\varphi)(g_2K), \quad (4.8)$$

for all  $g_1, g_2 \in G$  such that  $g_1g_2^{-1} \in J$ . If  $\delta^*$  is surjective, then there exists  $\varphi$  such that  $id_{\mathbb{Q}[G/K]} = \delta^*(\varphi)$ . But  $g_1K \neq g_2K$  for all  $g_1, g_2 \in G$  such that  $g_1g_2^{-1} \in J \setminus K \neq \emptyset$ , and the claim follows.

Finally, by Proposition 3.6 and Remark 3.4, one has

$$dH^1(G, \text{Bi}(G)) = \varinjlim_{\mathcal{CO}(G)} dH^1(G, \mathbb{Q}[G/U]), \quad (4.9)$$

where  $U$  is ranging over all compact open subgroups of  $G$ . Let  $\mathcal{CO}_K(G)$  be the set of all compact open subgroups of  $G$  contained in  $K$ . One has

$$dH^1(G, \text{Bi}(G)) = \varinjlim_{\mathcal{CO}_K(G)} dH^1(G, \mathbb{Q}[G/U]) \neq 0, \quad (4.10)$$

since the map

$$dH^1(\eta_{U,V}) : dH^1(G, \mathbb{Q}[G/U]) \rightarrow dH^1(G, \mathbb{Q}[G/V])$$

is injective for all compact open subgroups  $V \subseteq U \subseteq K$  of  $G$  (cf. Proof of [6, Proposition 4.7]) and  $dH^1(G, \mathbb{Q}[G/K]) \neq 0$ .  $\square$

In order to conclude the proof of Theorem A\* let us provide two Lemmas that concur to clarify the expected connection between number of ends and degree-1 cohomology.

Let  $K$  be a compact open subgroup of  $G$ . Following [9], a subset  $B \subset G/K$  is called an *almost  $(G, K)$ -invariant set* if the characteristic function  $\chi_B$  of  $B$  is an almost  $(G, K)$ -invariant function (cf. §3.3). In other words,  $B$  is an almost  $(G, K)$ -invariant set if  $gB =_a B$  (i.e. the symmetric difference is finite) for all  $g \in G$  and  $kB = B$  for all  $k \in K$ . Thus we reformulate a result of C. Bamford and M.J. Dunwoody (cf. [3, Lemma 1.1]) as follows.

**Lemma 4.4.** *Let  $G$  be a compactly generated t.d.l.c. group and let  $(K, S)$  be a generating pair. Then the  $\mathbb{Q}$ -vector space  $\mathcal{A}\text{Inv}_K(G, \mathbb{Q})$  of all almost  $(G, K)$ -invariant functions is generated by*

$$\{\chi_B \mid B \text{ almost } (G, K)\text{-invariant set}\}.$$

Note that if  $B$  is an almost  $(G, K)$ -invariant set, then its complement  $B^*$  is also an almost  $(G, K)$ -invariant set. An almost  $(G, K)$ -invariant set  $B \subset G/K$  is said to be *proper* if  $B, B^*$  are both infinite.

**Lemma 4.5.** *Let  $G$  be a compactly generated t.d.l.c. group and let  $(K, S)$  be a generating pair of  $G$ . If there exists a proper almost  $(G, K)$ -invariant set, then  $e(G) > 1$ .*

*Proof.* Let  $\Gamma = \Gamma(G, K, S)$  be the rough Cayley graph of  $G$  with respect to the generating pair  $(K, S)$ . If  $B \subset G/K$  is an infinite almost  $(G, K)$ -invariant set, in particular one has  $KB = B$  for all  $k \in K$ . Thus one defines

$$C_B = \{gK \in G/K \mid g^{-1}K \in B\} \subset V(\Gamma).$$

Clearly,  $C_B$  is infinite. Moreover, the set  $C_B$  has finite boundary. Indeed,

$$\bar{\delta}C_B = \{gsK \notin C_B \mid gK \in C_B, s \in S\} = \{s^{-1}g^{-1}K \notin B \mid g^{-1}K \in B, s \in S\}.$$

Rearranging, we have

$$\bar{\delta}C_B = \{gK \in G/K \mid \exists s \in S \text{ s.t. } \chi_B(gK) \neq s \cdot \chi_B(gK)\},$$

which is a finite set by the almost invariance of  $\chi_B$  and  $|S| < \infty$ . Clearly, if  $B$  is proper then  $C_B$  contains at least a cut of  $\Gamma$ . Thus Fact 2.3 completes the proof.  $\square$

*Proof of c)  $\Rightarrow$  a).* Since

$$dH^1(G, Bi(G)) = \varinjlim_{\mathcal{C}O(G)} dH^1(G, \mathbb{Q}[G/U]) \neq 0, \quad (4.11)$$

(cf. Proposition 3.6 and Remark 3.4), it suffices to prove that  $e(G) > 1$  if there exists a compact open subgroup  $K$  of  $G$  such that  $dH^1(G, \mathbb{Q}[G/K]) \neq 0$ .

Let  $K$  be such a subgroup. By Proposition 3.15, there is a non-trivial map  $d \in \mathcal{A}Inv_K(G, \mathbb{Q})$  which is neither constant on  $G/K$  nor almost zero. Since  $\mathcal{A}Inv_K(G, \mathbb{Q})$  is  $\mathbb{Q}$ -generated by the characteristic functions of the almost  $(G, K)$ -invariant sets of  $G$  (cf. Lemma 4.4), there exists an infinite almost  $(G, K)$ -invariant set  $B \subsetneq G/K$ . We claim that  $B$  is proper. Then the statement follows by Lemma 4.5.

Let us prove the claim. Set  $B^* = G \setminus B$  and  $d^* = g \cdot \chi_{B^*} - \chi_{B^*}$ . Clearly,  $d^* \in dH^1(G, \mathbb{Q}[G/K])$  and  $B^*$  is an infinite almost  $(G, K)$ -invariant set, i.e.  $B$  is proper.  $\square$

## 5. COMPACTLY PRESENTED T.D.L.C. GROUPS OF RATIONAL DISCRETE COHOMOLOGICAL DIMENSION ONE

Following [6], a *graph of profinite groups*  $(\mathcal{G}, \Lambda)$  based on the graph  $\Lambda$  consists of the following data:

- (G1) a profinite group  $\mathcal{G}_v$  for every vertex  $v \in V(\Lambda)$ ;
- (G2) a profinite group  $\mathcal{G}_e$  for every edge  $e \in E(\Lambda)$  satisfying  $\mathcal{G}_e = \mathcal{G}_e$ ;
- (G3) an open embedding  $\iota_e: \mathcal{G}_e \rightarrow \mathcal{G}_{t(e)}$  for every edge  $e \in E(\Lambda)$ .

The fundamental group of a graph of profinite groups carries naturally the structure of t.d.l.c. group. Indeed a neighbourhood basis of the identity is given by

$$\mathcal{B} := \{O \leq_{co} g\mathcal{G}_v g^{-1} \mid v \in V(\Lambda), g \in \pi_1(\mathcal{G}, \Lambda)\},$$

where  $O$  is a compact open subgroup of the vertex stabilizer  $g\mathcal{G}_v g^{-1}$ . We recall that a *generalized presentation* of a t.d.l.c. group  $G$  is a graph of profinite groups  $(\mathcal{G}, \Lambda)$  together with a continuous open surjective homomorphism

$$\phi: \pi_1(\mathcal{G}, \Lambda) \longrightarrow G, \quad (5.1)$$

such that  $\phi|_{\mathcal{G}_v}$  is injective for all  $v \in V(\Lambda)$ . In particular, every t.d.l.c. group  $G$  admits at least one generalized presentation  $(\mathcal{G}, \Lambda_0)$  based on a graph with a single vertex (cf. [6, Proposition 5.10]). A t.d.l.c. group  $G$

is said to be *compactly presented*, if there exists a generalized presentation  $((\mathcal{G}, \Lambda), \phi)$ , such that

- (i)  $\Lambda$  is a finite connected graph, and
- (ii)  $K = \ker(\phi)$  is a finitely generated as normal subgroup of the fundamental group  $\Pi = \pi_1(\mathcal{G}, \Lambda)$ .

Clearly, the fundamental group of a finite graph of profinite groups is a compactly presented t.d.l.c. group.

**Remark 5.1.** *The notion of being compactly presented we use here is equivalent to the usual one defined for compactly generated locally compact groups (cf. [2, Prop. 1.1.3]).*

Recall that a compactly generated t.d.l.c. group  $G$  is accessible if, and only if, it has an action on a tree  $\mathcal{T}$  such that:

- (A1) the number of orbits of  $G$  on the edges of  $\mathcal{T}$  is finite;
- (A2) the stabilizers of edges in  $\mathcal{T}$  are compact open subgroups of  $G$ ;
- (A3) every stabilizer of a vertex in  $\mathcal{T}$  is a compactly generated open subgroup of  $G$  and has at most one rough end.

**Theorem B.** *Let  $G$  be a t.d.l.c. group. Thus the following are equivalent:*

- (i)  $G$  is a compactly presented t.d.l.c. group with  $\text{cd}_{\mathbb{Q}}(G) \leq 1$ ,
- (ii)  $G$  is isomorphic to the fundamental group  $\pi_1(\mathcal{G}, \Lambda)$  of a finite graph of profinite groups  $(\mathcal{G}, \Lambda)$ .

*Proof.* Clearly, the fundamental group  $\Pi$  of a finite graph of profinite groups is a compactly presented t.d.l.c. group. Moreover  $\Pi$  acts on its universal covering tree without inversion of edges and with compact open vertex stabilizers. Then  $\text{cd}_{\mathbb{Q}}(\Pi) \leq 1$  (cf. (2.3) and Proposition 3.1).

Conversely, let  $G$  be a compactly presented t.d.l.c. group. By (a) of Proposition 3.2, if  $\text{cd}_{\mathbb{Q}}(G) = 0$ , then  $G$  is profinite and there is nothing to prove. Let  $\text{cd}_{\mathbb{Q}}(G) = 1$ . As  $G$  is compactly presented, by [7, Theorem 4.H.1]  $G$  is accessible. Thus  $G$  is acting on a tree  $\mathcal{T}$  with finitely many orbits on the set of edges and compact open edge stabilizers. Moreover every vertex stabilizer  $G_v$  is a compactly generated open subgroup of  $G$  with at most one end. By Theorem A\*, for all  $v \in V(\mathcal{T})$  one has  $\text{dH}^1(G_v, \text{Bi}(G_v)) = 0$ . By Propositions 3.6 and 3.2(b),  $G_v$  is of type  $\text{FP}_1$  with  $\text{cd}_{\mathbb{Q}}(G_v) \leq 1$ , i.e.,  $G_v$  is of type  $\text{FP}$  for any vertex  $v$ . Hence Proposition 3.8 implies  $\text{cd}_{\mathbb{Q}}(G_v) = 0$  for all  $v \in V(\mathcal{T})$ , i.e.,  $G_v$  is compact (cf. Proposition 3.2(a)). Finally, Bass-Serre's theory yields the claim.  $\square$

**Remark 5.2.** *Clearly Theorem B can be regarded as the analogue for t.d.l.c. groups of the Karrass-Pietrowski-Solitar theorem for virtually free groups, and in particular of Dunwoody's result [9, Thm. 1.1] on accessibility of discrete groups of cohomological dimension one.*

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